Solving

Linear Programming Problems

Graphically

Sem 5

Mathematics Paper 3

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Solving Linear Programming Problems Graphically

A linear programming problem involves constraints that contain inequalities. An **inequality** is denoted with familiar symbols, ≤, and ≥.

In order to have a linear programming problem, we must have:

- constraints in the form of Inequalities
- An **objective function**, that is, a function whose value we either want to to maximize or minimize .

Example 1

Railways offer 2^{nd} AC and first-class tickets. For the railways to be profitable, it must sell a minimum of 25 first-class tickets and a minimum of 40, 2^{nd} AC tickets. The railways makes a profit of Rs. 225 for each 2^{nd} AC ticket and Rs. 200 for each first-class ticket. At most, these two coaches have a capacity of 150 passengers. How many of each ticket should be sold in order to maximize profits?

Solution

We first identify the unknown quantities. We are asked to find the number of each ticket that should be sold. Since there are 2^{nd} AC and first-class tickets, we identify those as the unknowns. Let,

x = No. of 2^{nd} AC tickets

y = No. of first-class tickets

Next, we formulate the objective function. In this question the objective is to maximise the profit.

Profit for 2^{nd} AC tickets is Rs. 225, so the total profit for x tickets is 225x.

Profit for first-class tickets is Rs. 200, so the total profit for y tickets is 200y.

The total profit, P, is

$P = 225x + 200y$

We want to make the value of P as large as possible, provided the constraints are met. In this case, we have the following constraints:

- Sell at least 25 first-class tickets
- Sell at least 40 2^{nd} AC tickets
- Not more than 150 tickets can be sold

We now write the constraints:

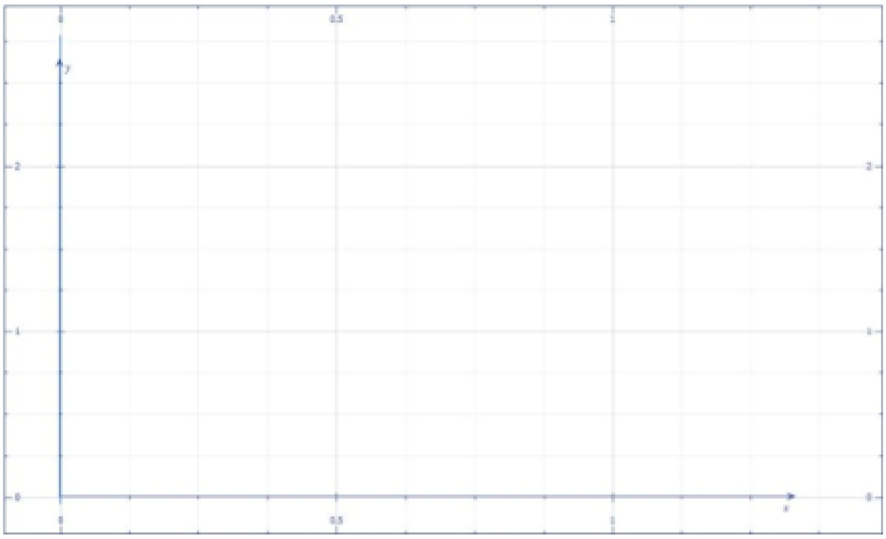
- At least 25 first-class tickets should be sold. That is, $y \geq 25$
- At least 40 2nd AC tickets should be sold. That is, $x \geq 40$
- The sum of first-class and 2nd AC tickets should be maximum150 . That is $x + y \leq 150$

Therefore, the objective function and the three mathematical constraints are:

Objective Function: $P = 225x + 200y$

Constraints: $y \geq 25$; $x \geq 40$; $x + y \leq 150$

We will now plot the constraints on the graph.



All plotting will be done in the first quadrant, since we cannot have negative tickets.

We will first assume the inequalities as equations, and plot, ie we have

$x = 25$

$y = 40$

$x + y = 150$

The first two equations are horizontal and vertical lines, respectively. To plot $x + y = 150$, it is preferable to find the horizontal and vertical intercepts.

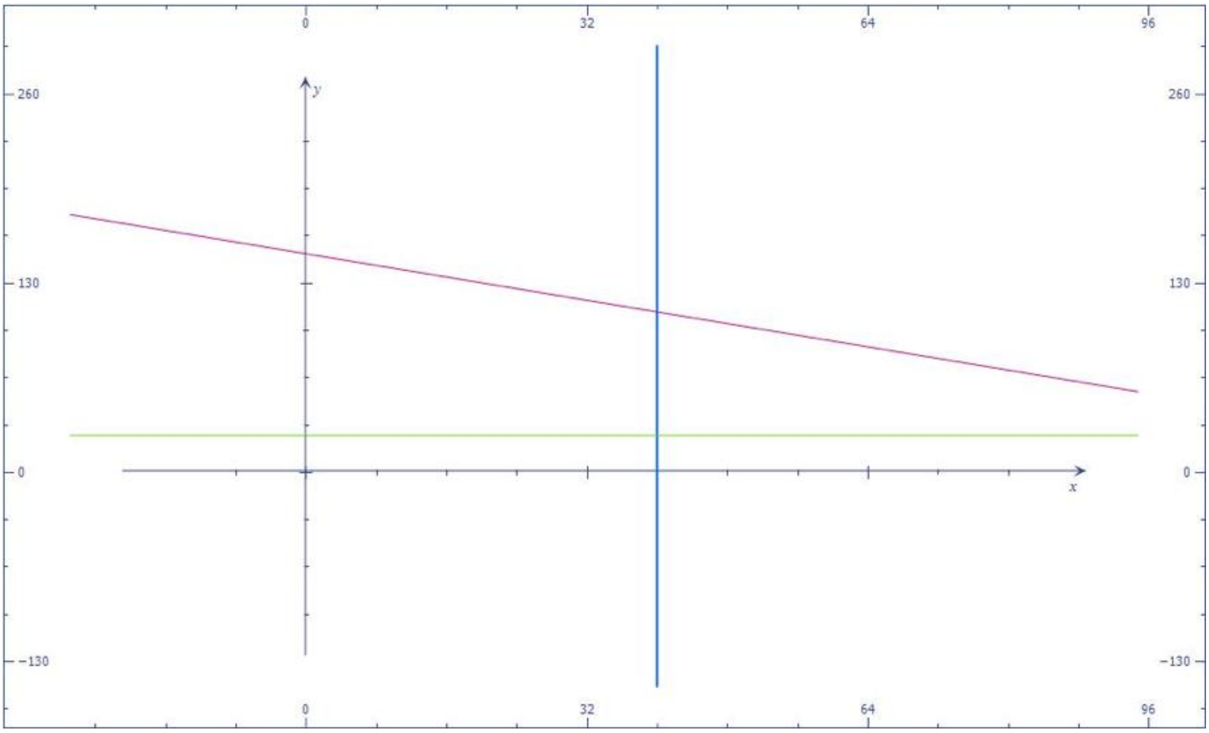
To find the vertical intercept, we let
$x = 0$:
$y = 150$

Giving us the point (0,150)

To find the horizontal intercept, we let
$y = 0$:
$x = 150$

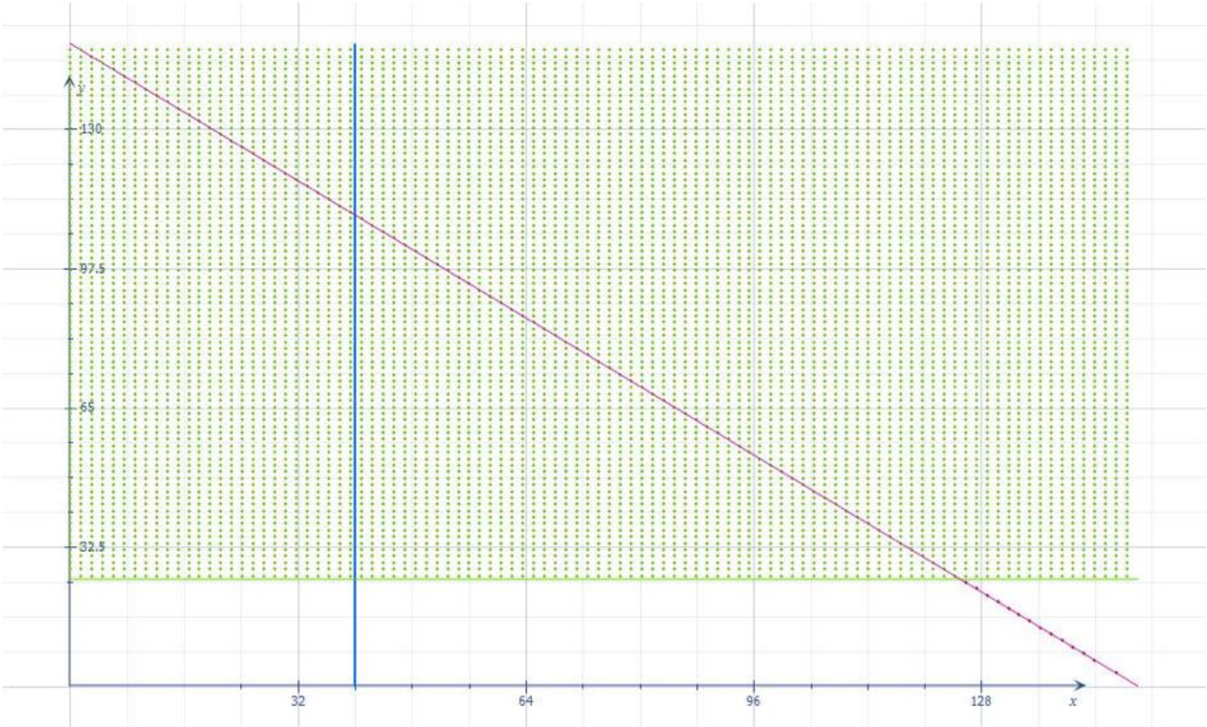
Giving us the point (150,0)

Plotting all three equations gives:



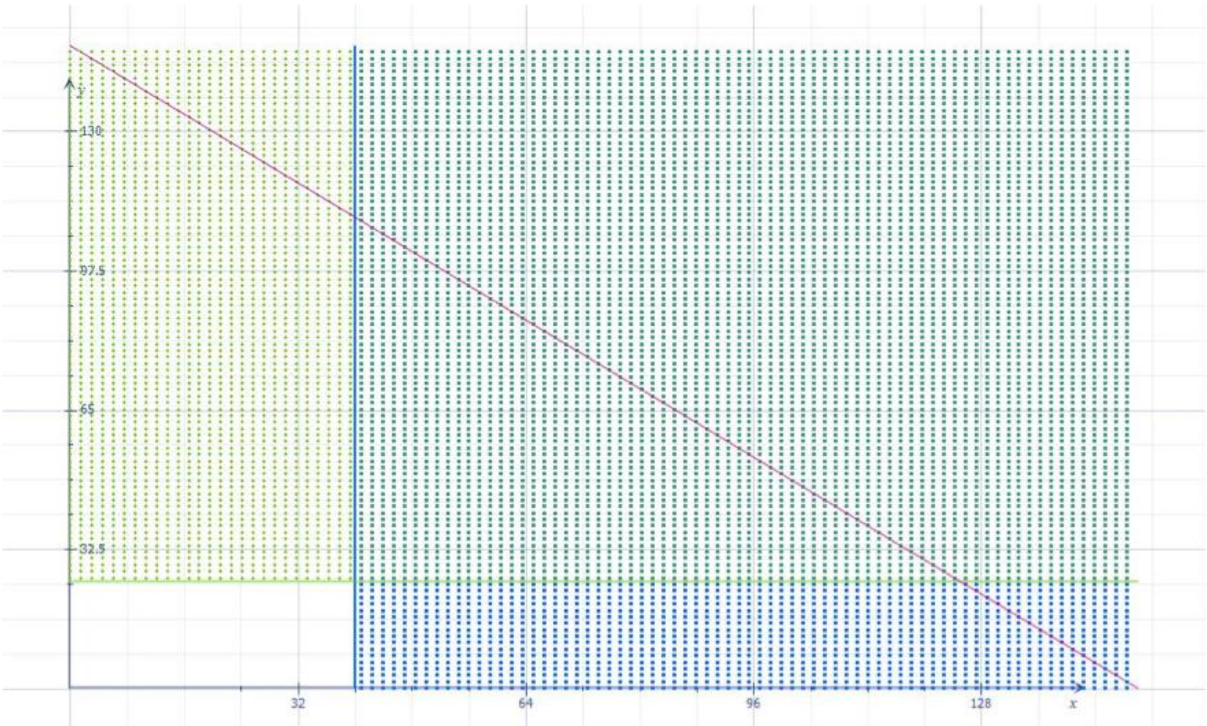
But actually we were given with inequalities not equations.

So let us think when is $y \geq 25$? Since this is a horizontal line running *through* a *y*-value of 25, anything above this line represents a value greater than 25. We denote this by shading above the line:



Thus we get that any point in the green shaded region satisfies the constraint that *y* ≥ 25.

Next, let us think about *x* ≥ 40. So we must shade to the right to get values more than 40:



The blue area satisfies the second constraint, but since we must satisfy *all* constraints, only the region that is green and blue will suffice.

Solving Linear Programming Problems Graphically

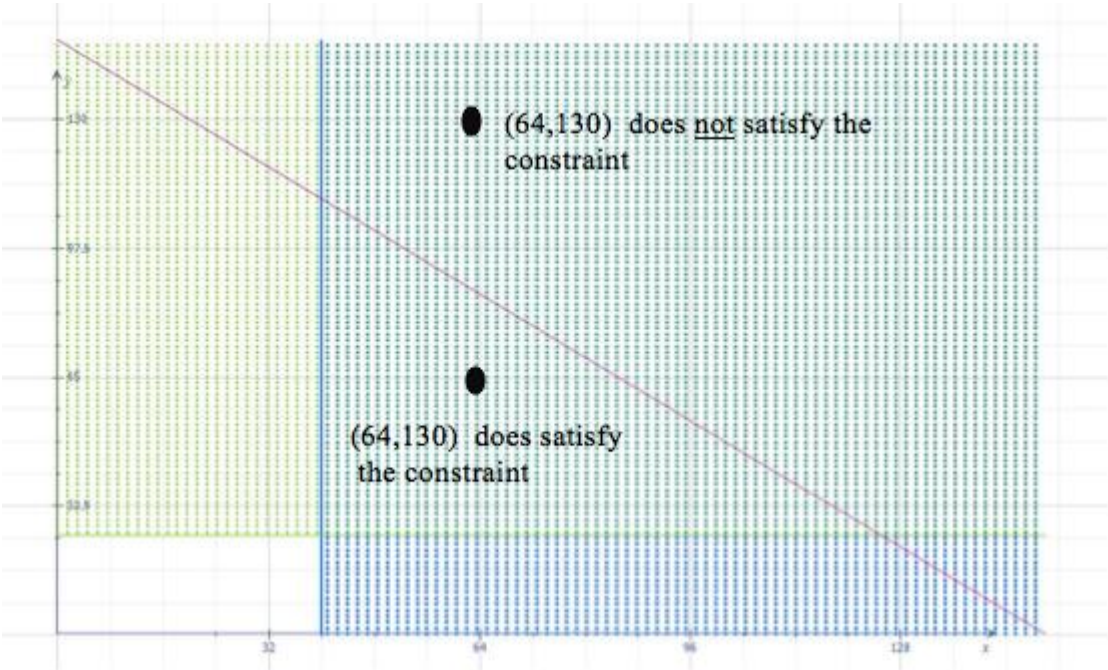
There is another constraint for consideration ie
x+ *y* ≤ 150. We have two options, either shade below or shade above the line x+y =150. Now to understand better we will, select an ordered pair above the line, such as (64, 130) gives:

64 + 130 ≥ 150 . We can see this does not satisfy the constraint *x*+ *y* ≤ 150

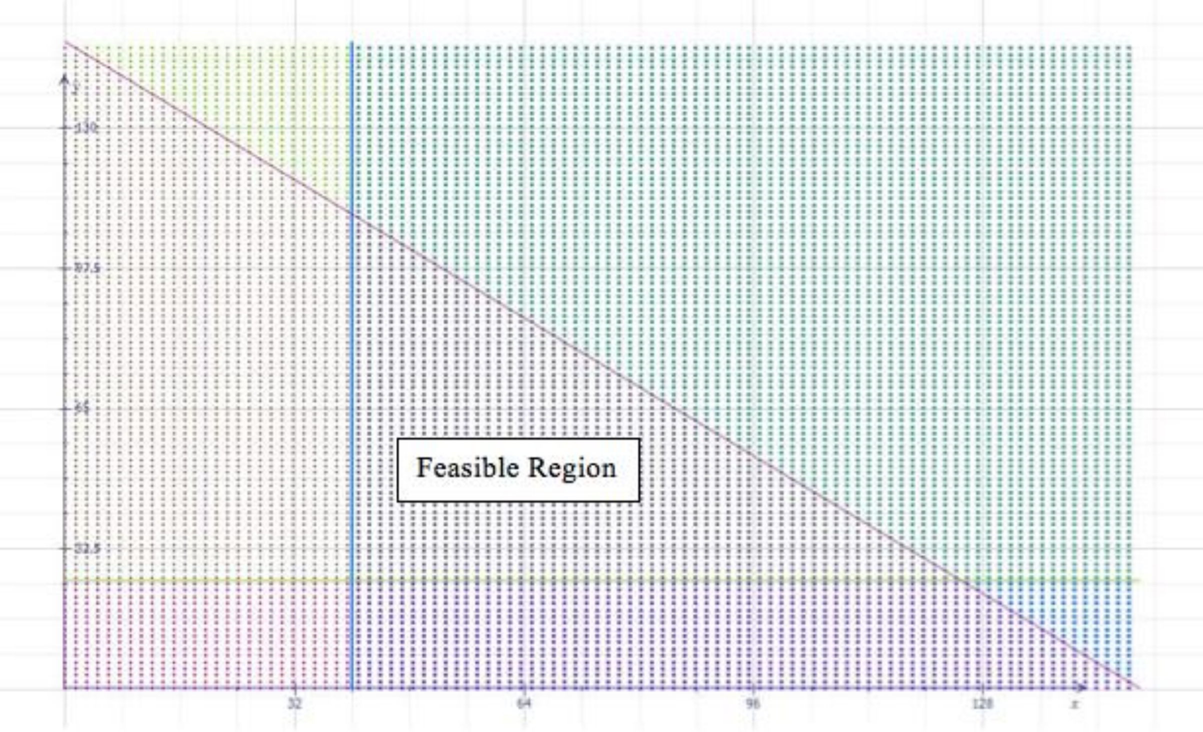
Next we consider the point (64, 65) below the line x+y =150. Putting this pair in yields the statement:

64 + 65 ≤150

Which is a true statement since 64+65 is 129, a value smaller than 150.



Therefore, we shade below the line:



The region in which the green, blue, and purple shadings intersect satisfies all three constraints. This region is known as the **feasible regions**, since this set of points is feasible, given all constraints. We can verify that a point chosen in this region satisfies all three constraints. For example, choosing (64, 65) gives:

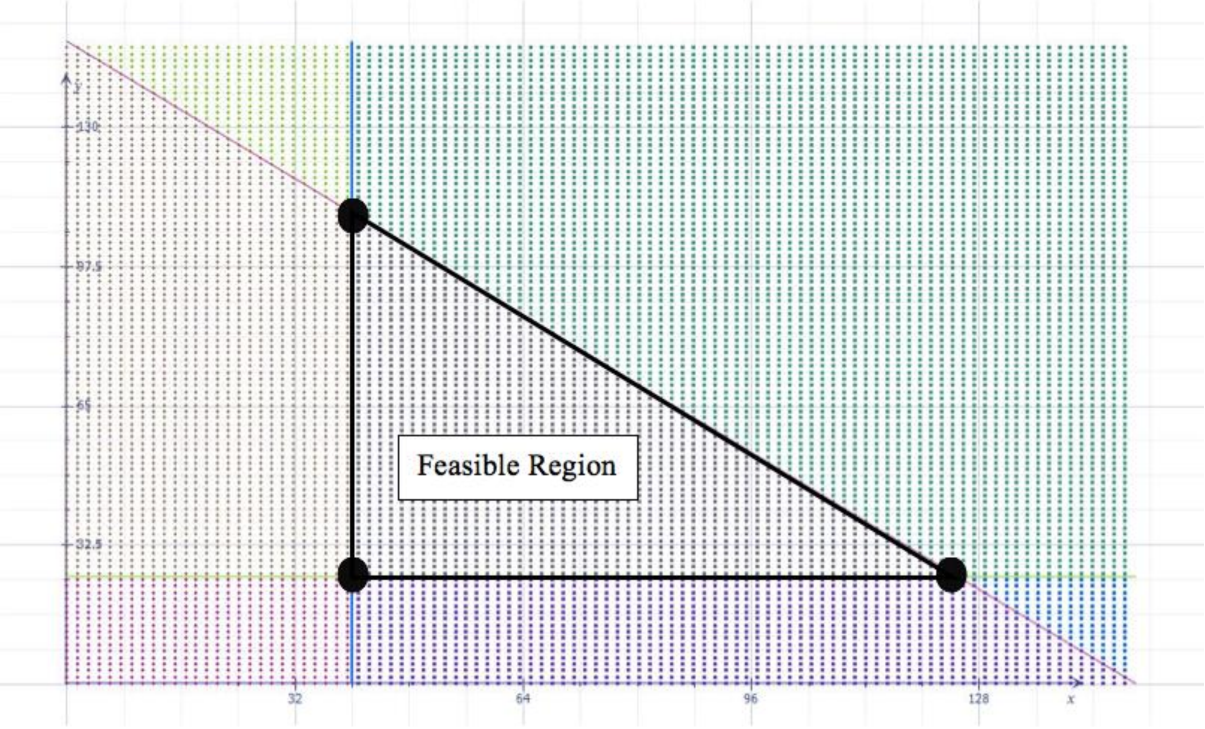
$64 \geq 40$ TRUE $\quad$ $(x \geq 40)$

$65 \geq 25$ TRUE $\quad$ $(y \geq 25)$

$64 + 65 \leq 150$ TRUE $\quad$ $(x+y \leq 150)$

So now we know the region having the solution but we do not know <u>which</u> point maximizes profit?

For this , first of all we define a new term: a **corner point** is a point that falls along the corner of a feasible region. In our situation, we have three corner points, shown on the graph as the solid black dots:



This is a bounded linear programming problem.

Fundamental Theorem of Linear Programming

1. If a solution exists to a bounded linear programming problem, then it occurs at one of the corner points.
2. If a feasible region is unbounded, then a maximum value for the objective function does not exist.
3. If a feasible region is unbounded, and the objective function has only positive coefficients, then a minimum value exist

This means we have to choose among three corner points. To verify the "winner," we must see which of these three points maximizes the objective function. To find the corner points as ordered pairs, we must solve three systems of two equations each:

System 1

$x = 40$

$x + y = 150$

System 2

$y = 25$

$x + y = 150$

System 3

$x = 40$

$y = 25$

We could decide to solve by using matrix equations, but these equations are all simple enough to solve by hand:

System 1

$(40) + y = 150$

$y = 110$

Point:(40,110)

System 2

$x + 25 = 150$

$x = 125$

Point: (125,25)

System 3

Point already given

Point: (40,25)

We test each of these three points in the objective function:

Point	Profit
(40,110)	225(40) + 200(110) = Rs.31,000
(125,25)	225(125) + 200(25) = Rs.33,125
(40,25)	225(40) + 200(25) = Rs.14,000

The second point, (125,25) maximizes profit. Therefore, we conclude that the railways should sell 125 2^{nd} AC tickets and 25 first-class tickets in order to maximize profits.

So we can summarise the procedure:

Solving a Linear Programming Problem Graphically

1. Define the variables to be optimized.
2. Write the objective function in words, then convert to mathematical equation
3. Write the constraints in words, then convert to mathematical inequalities
4. Graph the constraints as equations
5. Shade feasible regions by taking into account the inequality sign and its direction. If,

a)A vertical line

≤, then shade to the left

≥, then shade to the right

b) A horizontal line

≤, then shade below

≥, then shade above

c) A line with a non-zero, defined slope

≤, shade below

≥, shade above

6. Identify the corner points by solving systems of linear equations whose intersection represents a corner point.

7. Test all corner points in the objective function. The “winning” point is the point that optimizes the objective function (biggest if maximizing, smallest if minimizing)

There is one instance in which we must take great caution.

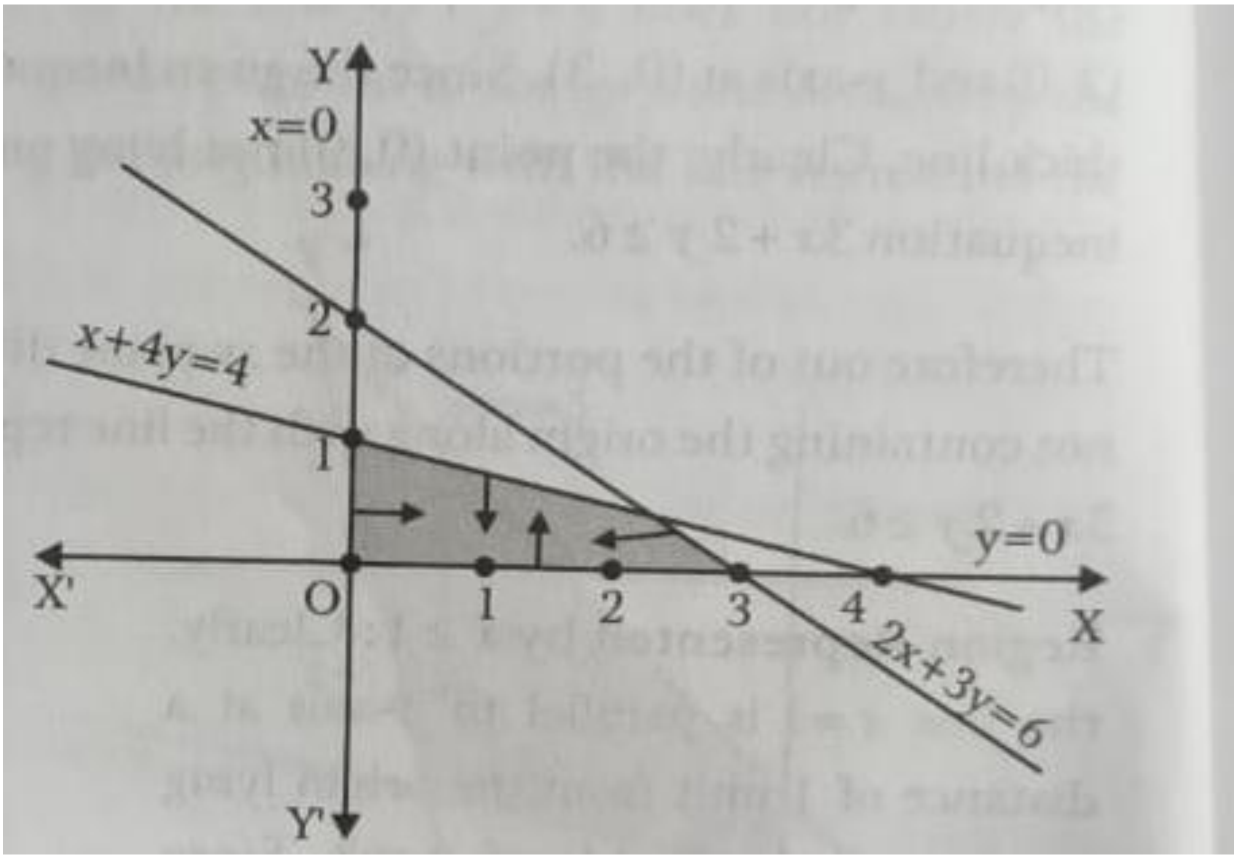
Changing the Inequality Sign

When multiplying/dividing any inequality by –1, the direction of the inequality should change.

Example 2

Draw the diagram of the solution set of the system of linear inequations:

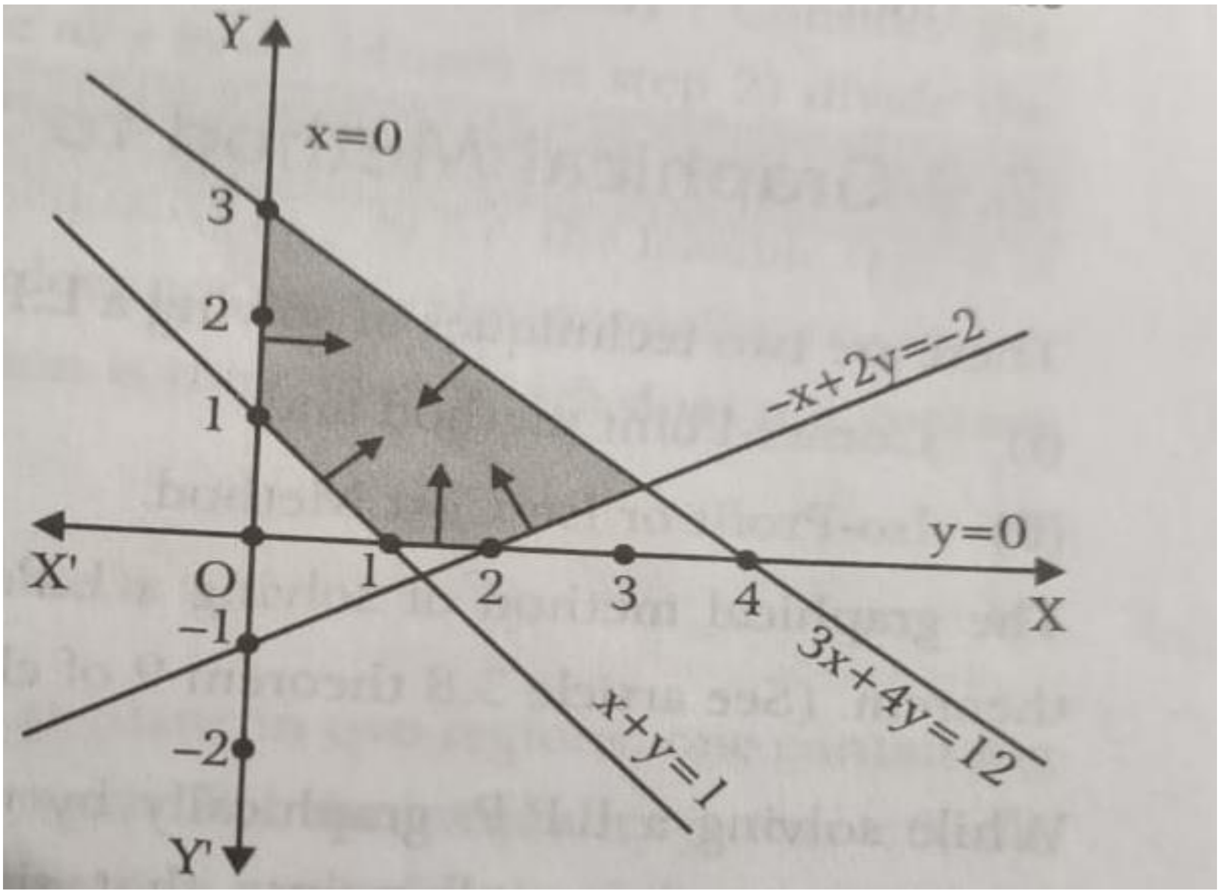
$2x+3y \leq 6$, $x+4y \leq 4$, $x \geq 0$, $y \geq 0$



Example 3

Exhibit graphically the solution set of the following system of linear inequations:

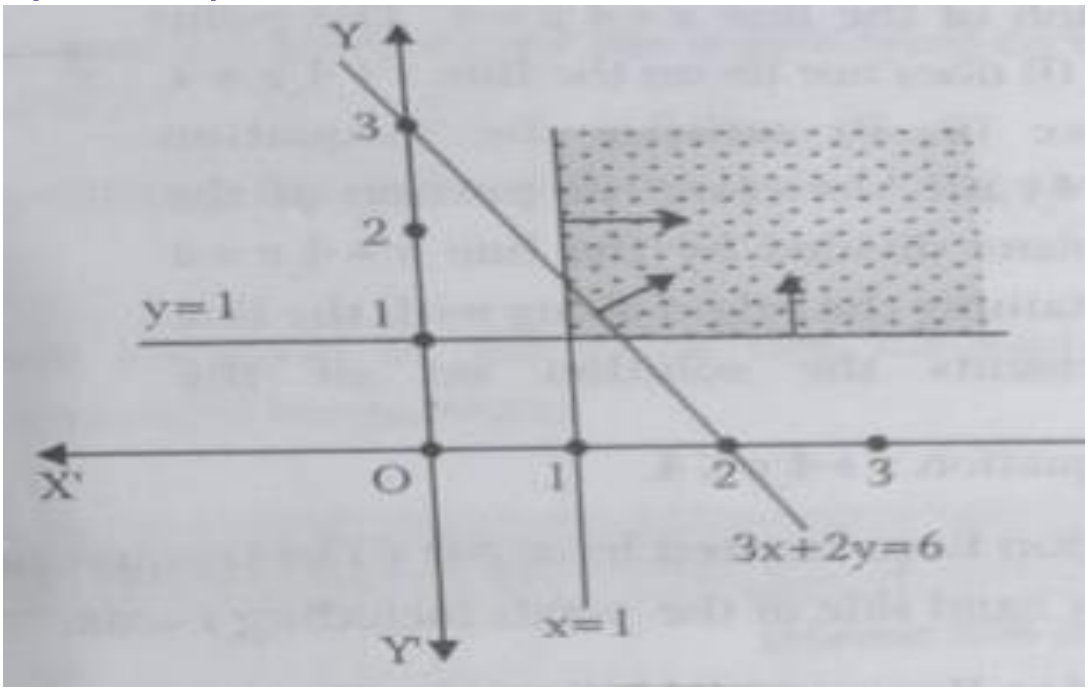
$x + y \geq 1$, $-3x - 4y \geq -12$, $-x + 2y \geq -2$, $x \geq 0$, $y \geq 0$



Example 4

Exhibit graphically the solution set of the linear inequations:

3x + 2y ≥ 6, x ≥ 1, y ≥ 1



Example 5

Draw the graph of the solution set of the inequations:

$x + y \le 5,\ 4x + y \ge 4,\ x + 5y \ge 5,\ x \le 4,\ y \le 3$