Department of Mathematics

ALGEBRA

PAPER-II

B.Sc.-II (Sem-III)

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Group Theory

Binary operation - Let G is a non - empty set . Any map

$$f:GxG \rightarrow G$$

is called binary operation on G or binary composition in G.

The image of the ordered pair (a,b) ∈ GxG under f is denoted by a f b Addittion +, multiplication x or (˙) are used as binary operations on a set.

Thus an operation which combines two elements of a set to give another element of the same set is called a binary operation or a binary composition .

Algebraic structure - A non - empty set together with one or more than one binary operation is called an algebraic structure.

Example; (R, + ˙) is an algebraic structure

Group:

Definition: A non - empty set G together with an operation o is called a group if the following conditions are satisfied;

(1) Closure axiom: ∀ a,b ∈ G⇒ aob ∈ G
(2) Associative law: (aob) oc =ao (boc), $\forall\ a, bc,$ ∈ G
(3) Existence of identity: There exists an element e ∈ G, called identity such that

$$aoe=eoa=a, \quad \forall\ a \in G$$

(4) Existence of inverse: Corresponding to every a∈ G, there exists a^{-1} ∈ G such that.

$$a^{-1}oa = aoa^{-1} = e$$

This $a^{-1}$is called inverse of a.

Instead of saying that G is a group w.r.t the operation o, we always say that (G,o) is a group.

Abelion group - A group (G,0) is called an abelian or commutative group if it satisfieds the axiom

aob=boa, for all a,b,∈ G

Example of Groups:

(1) The set of all real number with ordinary addition as law of composition .Here the number o is an identity and the inverse of a is a^{-1}
(2) The set of all positive real numbers under ordinary multiplication.
(3) The set of all complex number with addition as composition.

Finite and Infinite group- The group (G,0) is called a finite or infinite group according as number of elements of G are finite or infinite.

Order of group- The number of elements in a group (G,o) is defined as the order of the group and is denoted by o (G) An infinite group is said to be of infinite order.

Semi group- A non-empty set G together a binary operation is called a semi group if o is assouciative in G.

A group consisting of the identity element alone is called a trivial group , other are called non - trivial group.

General Properties of Group:

Theorem 1. Uniqueness of identity; The identity element of a group is unique in G.

e is an element ⇒ ee'=e'e=e'

e' is an element ⇒ ee'=e'e=e

⇒e=e'e=ee'=e'
Thus we get e=e'

This prove the required result.
Theorem 2. Cancellation laws- hold good in a group that is let us consider a,b,c are arbitrary element of a group G Then

ab=ac=⇒ b=c (Left cancellation law)

ba=ca=⇒ b=c (Right cancellation law)
Proof, let e be the identity element in a group G.
a,b,c ∈ G be arbitray.

ab=ac⇒ a^{-1} (ab)=a^{-1} (ac)

⇒ $(a^{-1}a)b = (a^{-1}a)c$ by associative law

⇒ eb=ec

⇒b=c
This prove the first result.

Again ba=ca⇒ $(ba)a^{-1} = (ca)a^{-1}$

⇒ $b(aa^{-1}) = c(aa^{-1})$ by associative law

⇒be=ce

⇒b=c
This proves the second result.
Theorem 3. Uniqueness of inverse - The inverse of each element of a group is unique.

Proof . If possible , let b and c be two inverse & of the same element a of a group G , so that

$$ba=ab=e \quad ------(1)$$

$$ca=ac=e \quad ------(2)$$

e being the identity in G.

From(1) and (2) , we ahve ba= e = ca

⇒ba=ca

⇒ b=c by right cancellation law

Hence the result.

<u>Theorem 4.</u> If the inverse of a is a^{-1}, then the inverse of a^{-1} is a it self that is a^{-1} $^{-1}$=a

Proof Let be the inverse of an element a of a group G. then

$$a^{-1}a = e$$

$$\Rightarrow (a^{-1})^{-1}(a^{-1}a) = (a^{-1})^{-1}e$$

$\Rightarrow [(a^{-1})^{-1}a^{-1}]a = (a^{-1})^{-1}$ by associative law

$$\Rightarrow ea = (a^{-1})^{-1}$$

$$\Rightarrow a = (a^{-1})^{-1}$$

Reference Publications:-

Krishna Publications

Prakashan Kendra Publications

Schyam Series for Algebra